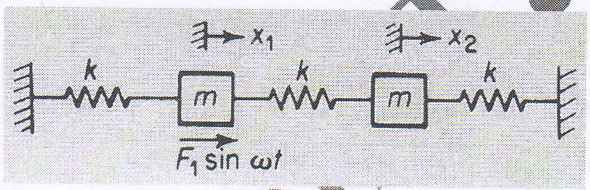
$$\text{adj}[A_{ij}] = [C_{ij}]^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Inverse matrix

$$[A_{ij}]^{-1} = \frac{\text{adj}[A_{ij}]}{|A_{ij}|}$$

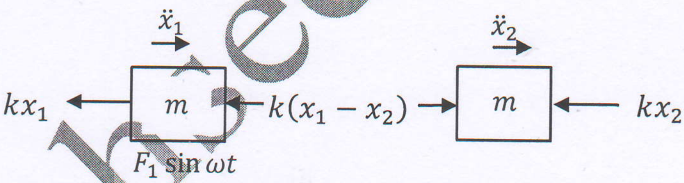
Note that $[A_{ij}]^{-1}[A_{ij}] = [\text{I}]$

Forced Harmonic Vibration of Two Degree Freedom Systems



Solution

Let $x_1(t) > x_2(t)$



From the free body diagram for left mass

$$\sum \mathbf{F} = m\ddot{x}_1 \;\; \Rightarrow \;\; -kx_1 - k(x_1 - x_2) + F_1 \sin\omega t = m\ddot{x}_1$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1 \sin\omega t \qquad \cdots\cdots\cdots (1)$$

From the free body diagram for right mass

$$\sum \mathbf{F} = m\ddot{x}_2 \;\; \Rightarrow \;\; -kx_2 + k(x_1 - x_2) = m\ddot{x}_2$$

$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \qquad \cdots\cdots\cdots (2)$$

Equations (1) and (2) are **equations of motion**

Similarly, Equations (1) and (2) have the matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \sin\omega t \\ 0 \end{Bmatrix} \qquad \cdots\cdots\cdots (*)$$

$$\boldsymbol{M}\ddot{\mathbf{x}} + \boldsymbol{K}\mathbf{x} = \boldsymbol{F}$$

in which

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \text{mass matrix} \ , \qquad \boldsymbol{F} = \begin{Bmatrix} F_1 \sin \omega t \\ 0 \end{Bmatrix} \ , \quad \boldsymbol{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} = \text{stiffness matrix}$$

To find the natural frequencies, let $\boldsymbol{F} = \boldsymbol{0}$

$$x_1(t) = A_1 e^{i\omega t}, \qquad x_2(t) = A_2 e^{i\omega t}$$

$$\ddot{x}_1(t) = -A_1 \omega^2 e^{i\omega t} \ , \qquad \ddot{x}_2(t) = -A_2 \omega^2 e^{i\omega t}$$

Then from the equations (1) and (2)

$$(2k - \omega^2 m)A_1 - kA_2 = 0 \qquad \cdots\cdots\cdots (3)$$

$$-kA_1 + (2k - \omega^2 m)A_2 = 0 \qquad \cdots\cdots\cdots (4)$$

By putting Equations (3) and (4) as a matrix form

substitution these Equations into Equation (1)

$$\begin{bmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - \omega^2 m) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

since $\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

then $$\begin{vmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - \omega^2 m) \end{vmatrix} = 0$$

which represents a quadratic equation in $\omega^2 = \lambda$ called the *characteristic equation*, or *frequency equation.*

$$\lambda^2 - \left(4\frac{k}{m}\right)\lambda + 3\left(\frac{k}{m}\right)^2 = 0$$

The two roots λ_1 and λ_2 of this equation are the *eigenvalues* of the system

$$\lambda_1 = \omega_1^2 = \frac{k}{m} \quad , \qquad \lambda_2 = \omega_2^2 = 3\frac{k}{m}$$

To find the **forced response** (since $F(t) = F_1 \sin \omega t$), let

$$x_1(t) = X_1 \sin \omega t, \qquad x_2(t) = X_2 \sin \omega t$$

$$\ddot{x}_1(t) = -X_1 \omega^2 \sin \omega t \ , \qquad \ddot{x}_2(t) = -X_2 \omega^2 \sin \omega t$$

Then from the equations (1) and (2)

$$(2k - \omega^2 m)X_1 - kX_2 = F_1 \qquad \cdots\cdots\cdots (5)$$

$$-kX_1 + (2k - \omega^2 m)X_2 = 0 \qquad \cdots\cdots\cdots (6)$$

By putting Equations (5) and (6) as a matrix form

$$\begin{bmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - \omega^2 m) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

Let

$$\begin{bmatrix} (2k-\omega^2 m) & -k \\ -k & (2k-\omega^2 m) \end{bmatrix} = [Z(\omega)]$$

$$[Z(\omega)]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [Z(\omega)]^{-1}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

where

$$[Z(\omega)]^{-1} = \frac{adj[Z(\omega)]}{|Z(\omega)|}$$

where

$$|Z(\omega)| = (2k-\omega^2 m)(2k-\omega^2 m) - k^2 = m^2\left(\omega^4 - 4\frac{k}{m}\omega^2 + 3k^2\right)$$

$$= m^2\left(\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k^2}{m^2}\right) = m^2\left(\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k}{m}\frac{k}{m}\right)$$

$$= m^2\left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - 3\frac{k}{m}\right) = m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)$$

$$\boxed{|Z(\omega)| = m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$adj[Z(\omega)] = \begin{bmatrix} (2k-\omega^2 m) & k \\ k & (2k-\omega^2 m) \end{bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{\begin{bmatrix} (2k-\omega^2 m) & k \\ k & (2k-\omega^2 m) \end{bmatrix}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

Note that :- when $\omega = \omega_1$ and when $\omega = \omega_2$ the denominator =0 (resonance)

$$X_1 = \frac{(2k-\omega^2 m)F_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

Vibration Absorber